Protocol Verification and State Space Methods

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Outline

1. Lecture 2: Introduction to model checking
2. Lecture 3a: Specification and model checking of Time Petri Nets and Timed Automata
3. Lecture 3b: CPN, modules, and data types
4. Lecture 4a: Parametric model checking for PN
5. Lecture 4b: Model checking CPN
6. Lecture 5a: Case studies using VerICS
7. Lecture 5a: Case studies using CPN Tools
Introduction to model checking

Outline

- Standard non-symbolic model checking algorithms for CTL and LTL.
- Partial order reductions for LTL_{¬X} and CTL_{¬X}.
- Introduction to symbolic model checking for CTL.
- BDD- and SAT-based model checking.
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Standard non-symbolic model checking algorithms for CTL and LTL.
Model checking problem

$M \models \varphi$

a Kripke model                      a modal formula
Syntax of CTL*

Syntax

S1. every member of $\mathcal{PV}$ is a state formula,
S2. if $\varphi$ and $\psi$ are state formulas, then so are $\neg \varphi$ and $\varphi \land \psi$,
S3. if $\varphi$ is a path formula, then $A\varphi$ and $E\varphi$, are state formulas,
P1. any state formula $\varphi$ is also a path formula,
P2. if $\varphi$, $\psi$ are path formulas, then so are $\varphi \land \psi$ and $\neg \varphi$,
P3. if $\varphi$, $\psi$ are path formulas, then so is $X\varphi$, $G\varphi$, and $\varphi U \psi$.

CTL* consists of the set of all state formulae.
Variety of sublogics of $\text{CTL}^*$

**Definition**

- $\text{LTL} \subseteq \text{CTL}^*$ is the fragment of $\text{CTL}^*$ in which all modal formulas are of the form $A\varphi$, where $\varphi$ does not contain the state modalities $A$, $E$.
- $\text{CTL} \subseteq \text{CTL}^*$ is the fragment of $\text{CTL}^*$ in which $A$, $E$, and the path modalities $U$ and $G$ may only appear paired: $AX$, $EX$, $AU$, $EU$, $AG$, and $EG$. 
Semantics of $\text{CTL}^*$

$M = (G, \iota, \Pi, V)$ - a model and $\pi = g_0a_0g_1 \cdots$ - an infinite path of $G$.

$\pi_i$ denotes the suffix $g_ia_ig_{i+1} \cdots$ of $\pi$

**S1.** $g \models q$ iff $q \in V(g)$, for $q \in PV$,

**S2.** $g \models \neg\varphi$ iff not $g \models \varphi$,

$g \models \varphi \land \psi$ iff $g \models \varphi$ and $g \models \psi$,

**S3.** $g \models A\varphi$ iff $\pi \models \varphi$ for every path $\pi$ starting at $g$,

$g \models E\varphi$ iff $\pi \models \varphi$ for some path $\pi$ starting at $g$,

**P1.** $\pi \models \varphi$ iff $g_0 \models \varphi$ for any state formula $\varphi$,

**P2.** $\pi \models \neg\varphi$ iff not $\pi \models \varphi$,

$\pi \models \varphi \land \psi$ iff $\pi \models \varphi$ and $\pi \models \psi$,

**P3.** $\pi \models X\varphi$ iff $\pi_1 \models \varphi$,

$\pi \models G\varphi$ iff $\pi_j \models \varphi$ for all $j \geq 0$,

$\pi \models \varphi U\psi$ iff there is an $i \geq 0$ such that $\pi_i \models \psi$ and $\pi_j \models \varphi$ for all $0 \leq j < i$. 
$M, \text{start} \models \mathsf{EG} \alpha$

$M, \text{start} \models \mathsf{E} \alpha \mathsf{U} \beta$
Semantics in Examples

\[ M, \text{start} \models EX\alpha \quad \text{and} \quad M, \text{start} \models EF\alpha \]
Model checking **CTL** by state labeling

**State labelling**

If we do not bother about the size of a model, then the simplest approach to **CTL** model checking, called **state labelling**, can be used.

**Algorithm**

We show a deterministic algorithm, based on state labelling, for determining whether a **CTL** formula $\varphi$ is true at a state $s \in S$ in a finite model $M = ((S, s^0, \rightarrow), V)$, of time complexity $O(|\varphi| \times (|S| + |\rightarrow|))$. 
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L.M. Kristensen and W. Penczek (CS)
Algorithm

The algorithm is designed so that when it finishes, each state $s$ of $M$ is labelled with the subformulas of $\varphi$ which are true at $s$.

- The algorithm operates in stages.
- The $i$-th stage handles all subformulas of $\varphi$ of length $i$ for $i \leq |\varphi|$.
- Thus, at the end of the last stage each state will be labelled with all subformulas of $\varphi$ which are true at it.
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Model checking \textbf{CTL} by state labeling

\textbf{CTL operators}

Each of the operators of \textbf{CTL} can be expressed in terms of the three operators \textit{EX}, \textit{EG}, and \textit{EU}.

\textbf{Five cases}

So, only 5 cases have to be considered, where \( \varphi \) is:

- \( \neg \psi \),
- \( \psi_1 \land \psi_2 \),
- \( \text{EX} \psi \),
- \( \text{E}(\psi_1 \text{U} \psi_2) \), or
- \( \text{EG} \psi \).

\textbf{Algorithm}

The algorithm is discussed for the last two cases only, as the others are straightforward.
Model checking \( \text{CTL} \) by state labeling

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Model checking CTL by state labeling

**Formula** \( \varphi = E(\psi_1 U \psi_2) \)

- The algorithm first finds all the states which are labelled with \( \psi_2 \) and labels them with \( \varphi \).
- It goes backwards using the relation \( \rightarrow^{-1} \) and finds all the states which can be reached by a path in which each state is labelled with \( \psi_1 \). All such states are labelled with \( \varphi \).

**Complexity**

This step requires time \( O(|S| + |\rightarrow|) \).
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\textbf{Formula} $\varphi = \text{EG} \psi$

- The graph $(S', \rightarrow')$ is constructed, where $S' = \{ s \in S \mid M, s \models \psi \}$ and $\rightarrow' = \rightarrow \cap (S' \times S')$.
- $(S', \rightarrow')$ is partitioned into strongly connected components and those states which belong to the components of size greater than 1 or with a self-loop are selected and labelled with $\varphi$.
- The algorithm goes backwards from these states using $\rightarrow^{-1}$ and finds all those states which can be reached by a path in which each state is labelled with $\psi$. It labels these states with $\varphi$.

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- $(S', \rightarrow')$ is partitioned into strongly connected components and those states which belong to the components of size greater than 1 or with a self-loop are selected and labelled with $\varphi$.
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Model checking CTL by state labeling

Example

Consider the model $M$ shown below and the CTL formula

$$\varphi = E(p_1 U(EGp_2)).$$

Figure: The model $M$. 
Model checking CTL by state labeling

Labelling M with $\varphi = E(p_1 U (EGp_2))$; $\varphi' = EGp_2$

(a) (b) (c) (d)
References and other approaches

Reference

The original state labelling algorithm for CTL was introduced by Clarke, Emerson, and Sistla in 1986.

Automata theoretic approaches

- By checking non-emptiness of the product of the automaton representing a system and an automaton accepting all the models of the negation of a formula, via ...
- A translation from CTL to alternating tree automata.
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Introduction to model checking for knowledge and time

Partial order reductions for $\text{LTL}_{-X}$ and $\text{CTL}_{-X}$. 
Networks of automata

Figure: TC composed of two trains and the controller
Algorithm DFS-POR

DFS-POR is used to compute paths of the reduced model. A stack represents the path $\pi = g_0 a_0 g_1 a_1 \cdots g_n$ currently being visited. For $g_n$, the following three operations are computed in a loop:

1. The set $en(g_n) \subseteq Act$ of enabled actions is identified and a subset $E(g_n) \subseteq en(g_n)$ of possible actions is heuristically selected.

2. For any action $a \in E(g_n)$ compute the successor state $g'$ of $g_n$ such that $g_n \xrightarrow{a} g'$, and add $g'$ to the stack. Recursively proceed to explore the submodel originating at $g'$.

3. Remove $g_n$ from the stack.
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Conditions for selection of $E(g)$

**Conditions**

**C1** No action $a \in Act \setminus E(g)$ that is dependent on an action in $E(g)$ can be executed before an action in $E(g)$ is executed.

**C2** On every cycle in the constructed state graph there is at least one node $g$ for which $E(g) = en(g)$.

**C3** Each action in $E(g)$ is invisible, i.e., does not change $V(g)$.

**C4** If $E(g) \neq en(g)$, then $E(g)$ is a singleton.
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## Conditions for selection of $E(g)$

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Correctness

**Theorem**

Let $M$ be a model and $M' \subseteq M$ be the reduced model generated by the **DFS-POR** algorithm. The following conditions hold:

a) If the choice of $E(g)$ is given by $C_1$, $C_2$, $C_3$, then $M \models \varphi$ iff $M' \models \varphi$, for any $\text{LTL}_{\neg X}$ formula $\varphi$.

b) If the choice of $E(g)$ is given by $C_1$, $C_2$, $C_3$ and $C_4$, then $M \models \varphi$ iff $M' \models \varphi$, for any $\text{CTL}^*_{\neg X}$ formula $\varphi$. 
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Experimental Results - Trains and controller (TC)

**Property**: if the train 1 is in the tunnel, then no other train is in the tunnel at the same time:

\[ AG(\text{in}_1 \rightarrow \bigwedge_{i=2}^{n} \neg \text{in}_i), \]

**State spaces**

- \( F(n) \) - the size of the full state space.
- \( R(n) \) - the size of the reduced state space.

\[ F(n) = c_n \times 2^{n+1}, \text{ for some } c_n > 1, \]

\[ R(n) = 3 + 4(n - 1). \]

The reduced state space is *exponentially smaller* than the original one, for both LTL\(^\_X\) and CTL\(^*_X\).
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Introduction to model checking

Introduction to symbolic model checking for CTL.
Fixed-point verification for CTL

**Introduction**
Symbolic and non-symbolic model checking methods can exploit the fixed point characterization of CTL formulas.

**Using fixpoints**
Labelling the states with the subformulas or computation of OBDD representation of a formula uses the standard algorithms for computing the least and the greatest fixpoints as follows.
Fixed-point verification for CTL

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Fixed points approach

**Axioms for CTL**

- \( \text{EG}\varphi \equiv \varphi \land \text{EXEG}\varphi \),
- \( \text{E}(\varphi \text{U}\psi) \equiv \psi \lor (\varphi \land \text{EX}(\text{E}(\varphi \text{U}\psi))) \),

**Fixed point characterization of CTL**

Let \([\varphi] = \{ s \in S \mid s \models \varphi \}\).

- \([\text{EG}\varphi] = [\varphi] \land [\text{EXEG}\varphi] \),
- \([\text{E}(\varphi \text{U}\psi)] = [\psi] \lor ([\varphi] \land [\text{EXE}(\varphi \text{U}\psi)]) \)

**Pre-set**

Let \( \text{pre}(X) = \{ s \in S \mid (\exists s' \in X) \, s \rightarrow s' \} \), for \( X \subseteq S \).

**Characterization**

- \([\text{EG}\varphi] = [\varphi] \land \text{pre}([\text{EG}\varphi]) \),
- \([\text{E}(\varphi \text{U}\psi)] = [\psi] \lor ([\varphi] \land \text{pre}([\text{E}(\varphi \text{U}\psi)])) \).
Fixed points approach

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Fixed point characterization of CTL

Let \( \llbracket \varphi \rrbracket = \{ s \in S \mid s \models \varphi \}. \)

- \( \llbracket \text{EG} \varphi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \text{EXEG} \varphi \rrbracket, \)
- \( \llbracket \text{E} (\varphi \text{U} \psi) \rrbracket = \llbracket \psi \rrbracket \cup (\llbracket \varphi \rrbracket \cap \llbracket \text{EXE} (\varphi \text{U} \psi) \rrbracket) \)

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- \( \llbracket \text{EG} \varphi \rrbracket = \llbracket \varphi \rrbracket \cap \text{pre} (\llbracket \text{EG} \varphi \rrbracket), \)
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## Fixed points approach

### Axioms for CTL

- \( \text{EG} \varphi \equiv \varphi \land \text{EXEG}\varphi \),
- \( \text{E}(\varphi \cup \psi) \equiv \psi \lor (\varphi \land \text{EX}(\text{E}(\varphi \cup \psi))) \),

### Fixed point characterization of CTL

Let \([\varphi] = \{ s \in S \mid s \models \varphi \} \).

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### Pre-set

Let \( \text{pre}(X) = \{ s \in S \mid (\exists s' \in X) s \to s' \} \), for \( X \subseteq S \).

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Fixed points

Functions

Define two functions on $2^S$, which fixed points are equal to respectively $[\text{EG}\varphi]$ and $[\text{E}(\varphi U \psi)]$.

1. $\tau_{\text{EG}\varphi}(X) = [\varphi] \cap \text{pre}(X)$,
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Computing fixed points

- $[\text{EG}\varphi]$ is the greatest fixpoint of $\tau_{\text{EG}\varphi}(X)$, so it can be computed as $\tau_{\text{EG}\varphi}^k(S)$ for some finite $k$.
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The above characterization can be now used for defining a model checking algorithm $\text{mchk}$ for the formulas of CTL.
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Algorithm

Model checking algorithm based of fixpoint characterization

\[
mchk(M, \varphi) \{
\text{if } \varphi \in PV, \text{ then return } V^{-1}(\varphi), \\
\text{if } \varphi = \neg \psi, \text{ then return } S \setminus mchk(M, \psi), \\
\text{if } \varphi = \varphi_1 \lor \varphi_2, \text{ then return } mchk(M, \varphi_1) \cup mchk(M, \varphi_2), \\
\text{if } \varphi = \text{EX}\psi, \text{ then return } mchk_{\text{EX}}(M, \psi), \\
\text{if } \varphi = \text{EG}\psi, \text{ then return } mchk_{\text{EG}}(M, \psi), \\
\text{if } \varphi = E(\psi_1 U \psi_2), \text{ then return } mchk_{\text{EU}}(M, \psi_1, \psi_2), \\
\}
\]
Algorithm: Model checking procedures

\[ mchk_{EX}(M, \psi) \{ \]
\[ X := mchk(M, \psi); \]
\[ Y := pre(X); \]
\[ return \ Y \}; \]

\[ mchk_{EG}(M, \psi) \{ \]
\[ X := mchk(M, \psi); \]
\[ Y := S; \]
\[ Z := \emptyset; \]
\[ while (Z \neq Y) \{ \]
\[ Z := Y; \]
\[ Y := X \cap pre(Y) \} \]
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OBDD

OBDD (Ordered Binary Decision Diagrams) are used for succinct representation of Boolean functions. Consider a Boolean function:

\[ f : \{0, 1\}^n \rightarrow \{0, 1\} \]

A function can be represented by the results of all the valuations of some propositional formula over \( n \) propositional variables.

Example

For example the function \( f(x_1, x_2) = x_1 \ast x_2 \) is represented by the formula \( p_1 \land p_2 \). Each Boolean function can be represented by an OBDD.
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Introduction to OBDDs

**Figure:** BDD representing the boolean function $f$ (source: Wikipedia)
**Introduction to OBDDs**

**Figure:** Canonical OBDD representing the boolean function $f$ (source: Wikipedia)
Introduction to OBDDs

Figure: OBDD with a bad variable ordering (source: Wikipedia)
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The following operations can be implemented by polynomial-time graph manipulation algorithms:

- disjunction,
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OBDD-based model checking for CTLK

Fixed point algorithms on OBDD

The algorithms computing for each formula \( \varphi \) the set of states \( \mathbb{[}\varphi]\) in which \( \varphi \) holds, can operate on the OBDD representations of the states.

Encoding

This requires to encode the states and the transition relation of a model \( M \) by propositional formulas, and then to represent these formulas by OBDDs.

Model checking

\( M, s^0 \models \varphi \) is translated to checking whether \( s^0 \in \mathbb{[}\varphi]\).

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\text{OBDD}\{\{s^0\}\} \land \text{OBDD}(\mathbb{[}\varphi]\}) \not= \text{OBDD}(\emptyset)
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\( \text{OBDD}(S) \) denotes the OBDD representing the set of states \( S \).
OBBD-based model checking for CTLK

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SAT solvers

**Complexity**

- **Problem:** is a propositional formula satisfiable?
- Theoretical complexity: NP-complete (Cook, 1971),
- Practical and efficient SAT solvers: only in the last decade,
- Many competing algorithms: DPLL scheme is the most successful,
- A general idea: search efficiently for a satisfying assignment.

**Efficiency**

A SAT-solver is a heuristics only, but it can be very “clever”. Modern SAT-solvers can decide formulas composed of hundreds of thousands of propositional variables in a reasonable time.
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- Efficient **data** representation,
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- CNF: conjunctive normal form, conjunction of disjunctions of literals,

\[(\neg p_1) \land (p_1 \lor p_4 \lor \neg p_5) \land (\neg p_2 \lor p_3) \land (p_4 \lor p_5)\]
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Two fragments of CTL

Syntax of ACTL

The logic ACTL is the restriction of CTL such that it consists of the formulas of the form: $AX\alpha$, $A(\alpha U \beta)$, $AG\alpha$.

So, the formulas are only in the universal form (no negation applied to modalities).

Syntax of ECTL

The language of ECTL is defined as $\{\neg \varphi \mid \varphi \in ACTL\}$

After 'pushing' negation down the formula, we have the formulas only in the existential form (no negation applied to modalities): $EX\alpha$, $E(\alpha U \beta)$, $EG\alpha$. 
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Idea of Bounded Model Checking BMC

**BMC:** to prove that an ECTL formula holds or that an ACTL formula does not hold in $M$

1. If $\psi \in \text{ACTL}$, take negation $\varphi := \neg \psi$
2. If $\psi \in \text{ECTL}$, $\varphi := \psi$
3. Take a fragment $M'$ of the model $M$ (preserving $\varphi$, i.e., $M' \models \varphi$ implies $M \models \varphi$),
4. Translate $M' \models \varphi$ to a Boolean formula $[M'] \land [\varphi]_{M'}$
   $(M' \models \varphi$ iff $[M'] \land [\varphi]_{M'}$ is satisfiable),
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If $[M'] \land [\varphi]_{M'}$ is satisfiable, then $M \models \varphi$. 
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4. Translate $M' \models \varphi$ to a Boolean formula $[M'] \land [\varphi]_{M'}$, $(M' \models \varphi$ iff $[M'] \land [\varphi]_{M'}$ is satisfiable),
5. Check the satisfiability of $[M'] \land [\varphi]_{M'}$ with SAT-solvers.

**Conclusion:**
If $[M'] \land [\varphi]_{M'}$ is satisfiable, then $M \models \varphi$. 
Idea of Bounded Model Checking BMC

**BMC**: to prove that an ECTL formula holds or that an ACTL formula does not hold in $M$

1. If $\psi \in$ ACTL, take negation $\phi := \neg \psi$
2. If $\psi \in$ ECTL, $\phi := \psi$
3. Take a fragment $M'$ of the model $M$ (preserving $\phi$, i.e., $M' \models \phi$ implies $M \models \phi$),
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Idea of Bounded Model Checking BMC

**BMC**

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1. If $\psi \in \text{ACTL}$, take negation $\varphi := \neg \psi$.
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**Conclusion:**

If $[M'] \land [\varphi]_{M'}$ is satisfiable, then $M \models \varphi$. 
BMC for an ECTL formula $\varphi$

- Let $\varphi$ be an ECTL formula,
- Iterate for $k := 1$ to $|M|$,
- Select the $k$–model $M_k$ (of the paths of length $k$),
- Select the $f_k(\varphi)$-submodels of $M_k$ (of $f_k(\varphi)$ paths),
- Translate the transition relation of the $k$–paths of $M_k$ to a propositional formula $[M^{\varphi,\ell}]_k$,
- Translate $\varphi$ over all the $f_k(\varphi)$-submodels to a propositional formula $[\varphi]_{M_k}$,
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BMC for an **ECTL** formula $\varphi$

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Function $f_k$

Define the function $f_k : \text{ECTL} \rightarrow \mathbb{N}$ as follows:

- $f_k(p) = f_k(\neg p) = 0$, where $p \in \mathcal{P}\mathcal{V}$,
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Intuition

The function $f_k(\alpha)$ computes the number of symbolic paths (sufficient) to represent submodels of $M_k$ in the propositional translation of $\alpha$. 
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The function $f_k(\alpha)$ computes the number of symbolic paths (sufficient) to represent submodels of $M_k$ in the propositional translation of $\alpha$. 
Let $M = (K, \mathcal{V})$ be a model and $k \in \mathbb{N}_+$. The $k$-model for $M$ is a structure

$$M_k = ((G, P_k, \iota), \mathcal{V}),$$

where

- $G$ - a set of the global states,
- $P_k$ is the set of all the paths of $M$ of length $k$,
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Bounded semantics for ECTLpK

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\[ s \models \text{EX}\alpha \iff \exists \pi \in P_k (\pi(0) = s \text{ and } \pi(1) \models \alpha), \]
\[ s \models \text{EG}\alpha \iff \exists \pi \in P_k (\pi(0) = s \text{ and } \forall 0 \leq j \leq k \pi(j) \models \alpha \land \text{loop}(\pi) \neq \emptyset), \]
\[ s \models E(\alpha U \beta) \iff (\exists \pi \in P_k) (\pi(0) = s \text{ and } \exists 0 \leq j \leq k (\pi(j) \models \beta \text{ and } \forall 0 \leq i < j \pi(i) \models \alpha)). \]

Intuition

The bounded semantics for \( s \models \text{EG}\alpha \) says that there is a \( k \)-path \( \pi \), which starts at \( s \), all its states satisfy \( \alpha \) and \( \pi \) is a loop, which means that one of the states of \( \pi \) is a \( \rightarrow \)-successor of \( \pi(k) \). \text{loop}(\pi) \) returns the indeces of such states.
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Main references and next lecture

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Next lecture

Specification and model checking of Time Petri Nets and Timed Automata.
Main references and next lecture

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Next lecture

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Thank you