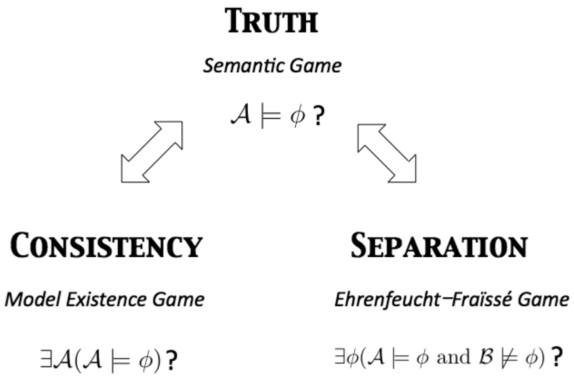

Figure 1.1 The Strategic Balance of Logic.

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3
Games

3.1 Introduction

In this first part we march through the mathematical details of zero-sum two-person games of perfect information in order to be well prepared for the introduction of the three games of the Strategic Balance of Logic (see Figure 1.1) in the subsequent parts of the book. Games are useful as intuitive guides in proofs and constructions but it is also important to know how to make the intuitive arguments and concepts mathematically exact.

3.2 Two-Person Games of Perfect Information

Two-person games of perfect information are like chess: two players set their wits against each other with no role for chance. One wins and the other loses. Everything is out in the open, and the winner wins simply by having a better strategy than the loser.

A Preliminary Example: Nim

In the game of Nim, if it is simplified to the extreme, there are two players **I** and **II** and a pile of six identical tokens. During each round of the game player **I** first removes one or two tokens from the top of the pile and then player **II** does the same, if any tokens are left. Obviously there can be at most three rounds. The player who removes the last token wins and the other one loses.

The game of Figure 3.1 is an example of a zero-sum two-person game of perfect information. It is zero-sum because the victory of one player is the loss of the other. It is of perfect information because both players know what the other player has played. A moment's reflection reveals that player **II** has a way

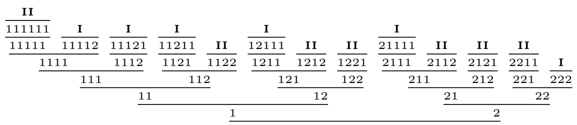

Figure 3.3

cabulary $L = \{W\}$, where W is a four-place predicate symbol. Let $\mathcal{M}$ be an L-structure[2] with $M = \{1, 2\}$ and

$$W^{\mathcal{M}} = \{(a_0, b_0, a_1, b_1) \in M^4 : a_0 + b_0 + a_1 + b_1 = 6\}.$$

Now we have just proved

$$\mathcal{M} \models \forall x_0 \exists y_0 \forall x_1 \exists y_1 W(x_0, y_0, x_1, y_1). \tag{3.1}$$

Conversely, if $\mathcal{M}$ is an arbitrary L-structure, condition (3.1) defines *some* game, maybe not a very interesting one but a game nonetheless: Player **I** picks an element $a_0 \in M$, then player **II** picks an element $b_0 \in M$. Then the same is repeated: player **I** picks an element $a_1 \in M$, then player **II** picks an element $b_1 \in M$. After this player **II** is declared the winner if $(a_0, b_0, a_1, b_1) \in W^{\mathcal{M}}$, and otherwise player **I** is the winner. By varying the structure $\mathcal{M}$ we can model in this way various two-person two-round games of perfect information. This gives a first hint of the connection between games and logic.

Games – a more general formulation

Above we saw an example of a two-person game of perfect information. This concept is fundamental in this book. In general, the simplest formulation of such a game is as follows (see Figure 3.4): There are two players[3] **I** and **II**, a domain A, and a natural number n representing the length of the game. Player **I** starts the game by choosing some element $x_0 \in A$. Then player **II** chooses $y_0 \in A$. After x_i and y_i have been played, and $i + 1 < n$, player **I** chooses $x_{i+1} \in A$ and then player **II** chooses $y_{i+1} \in A$. After n rounds the game ends. To decide who wins we fix beforehand a set $W \subseteq A^{2n}$ of sequences

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \tag{3.2}$$

[2] For the definition of an L-structure see Definition 5.1.
[3] There are various names in the literature for player **I** and **II**, such as player **I** and player **II**, spoiler and duplicator, Nature and myself, or Abelard and Eloise.

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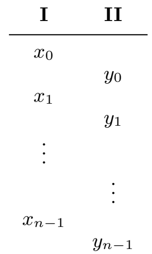

Figure 3.4 A game.

and declare that player **II** wins the game if the sequence formed during the game is in W; otherwise player **I** wins. We denote this game by $\mathcal{G}_n(A, W)$. For example, if $W = \emptyset$, player **II** cannot possibly win, and if $W = A^{2n}$, player **I** cannot possibly win. If W is a set of sequences $(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ where $x_0 = x_1$ and if moreover A has at least two elements, then **II** could not possibly win, as she cannot prevent player **I** from playing x_0 and x_1 differently. On the other hand, W could be the set of all sequences (3.2) such that $y_0 = y_1$. Then **II** can always win because all she has to do during the game is make sure that she chooses y_0 and y_1 to be the same element.

If player **II** has a way of playing that guarantees a sure win, i.e. the opponent **I** loses whatever moves he makes, we say that player **II** has a winning strategy in the game. Likewise, if player **I** has a way of playing that guarantees a sure win, i.e. player **II** loses whatever moves she makes, we say that player **I** has a winning strategy in the game. To make intuitive concepts, such as "way of playing" more exact in the next chapter we define the basic concepts of game theory in a purely mathematical way.

Example 3.1 The game of Nim presented in the previous chapter is in the present notation $\mathcal{G}_3(\{1, 2\}, W)$, where

$$W = \left\{ (a_0, b_0, a_1, b_1, a_2, b_2) \in \{1, 2\}^6 : \sum_{i=0}^{n} (a_i + b_i) = 6 \text{ for some } n \leq 2 \right\}.$$

We allow three rounds as theoretically the players could play three rounds even if player **II** can force a win in two rounds.

Example 3.2 Consider the following game on a set A of integers:

Example 3.5 The following game has no moves:

I	**II**

If $W = \{\emptyset\}$, player **II** is the winner. If $W = \emptyset$, player **I** is the winner. So this is a game with 0 rounds. In practice one of the players would find these games unfair as he or she loses without even having a chance to make a move. It is like being invited to play a game of chess starting in a position where you are already in check-mate.

3.3 The Mathematical Concept of Game

Let A be an arbitrary set and n a natural number. Let $W \subseteq A^{2n}$. We redefine the game

$$\mathcal{G}_n(A, W)$$

in a purely mathematical way. Let us fix two players **I** and **II**. A *play* of one of the players is any sequence $\bar{x} = (x_0, \ldots, x_{n-1})$ of elements of A. A sequence

$$(\bar{x}; \bar{y}) = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

of elements of A is called a *play* (of $\mathcal{G}_n(A, W)$). So we have defined the concept of play without any reference to playing the game as an act. The play $(\bar{x}; \bar{y})$ is a *win for* player **II** if

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in W$$

and otherwise a *win for* player **I**.

Example 3.6 Let us consider the game of chess in this mathematical framework. We modify the game so that the number of rounds is for simplicity exactly n and Black wins a draw, i.e. if neither player has check-mated the other player during those up to n rounds. If a check-mate is reached the rest of the n-round game is of course irrelevant and we can think that the game is finished with "dummy" moves. Let A be the set of all possible positions, i.e. configurations of the pieces on the board. A play $\bar{x}$ of **I** (White) is the sequence of positions where White has just moved. A play $\bar{y}$ of **II** is the sequence of positions where Black has just moved. We let W be the set of plays $(\bar{x}; \bar{y})$, where either White has not obeyed the rules, or Black has obeyed the rules and White has not check-mated Black. With the said modifications, chess is just the game $\mathcal{G}_n(A, W)$ with White playing as player **I** and Black playing as player **II**.

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A *strategy* of player **I** in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\sigma = (\sigma_0, \ldots, \sigma_{n-1})$$

of functions $\sigma_i : A^i \to A$. We say that player **I** has *used the strategy* σ *in the play* $(\bar{x}; \bar{y})$ if for all $0 < i < n$:

$$x_i = \sigma_i(y_0, \ldots, y_{i-1})$$

and

$$x_0 = \sigma_0.$$

The strategy σ of player **I** is a *winning strategy*, if every play where **I** has used σ is a win for player **I**. Note that the strategy depends only on the opponent's moves. It is tacitly assumed that when the function σ_{i+1} is used to determine x_{i+1}, the previous functions $\sigma_0, \ldots, \sigma_i$ were used to determine the previous moves $x_0, \ldots, x_n$. Thus a strategy σ is a winning strategy because of the concerted effect of all the functions $\sigma_0, \ldots, \sigma_{n-1}$.

A *strategy* of player **II** in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\tau = (\tau_0, \ldots, \tau_{n-1})$$

of functions $\tau_i : A^{i+1} \to A$. We say that player **II** has *used the strategy* τ *in the play* $(\bar{x}; \bar{y})$ if for all $i < n$:

$$y_i = \tau_i(x_0, \ldots, x_i).$$

The strategy τ of player **II** is a *winning strategy*, if every play where player **II** has used τ is a win for player **II**. A player who has a winning strategy in $\mathcal{G}_n(A, W)$ is said to *win the game* $\mathcal{G}_n(A, W)$.

3.4 Game Positions

A *position* of the game $\mathcal{G}_n(A, W)$ is any initial segment

$$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$$

of a play $(\bar{x}; \bar{y})$, where $i \leq n$. Positions have a natural ordering: a position p' extends a position p, if p is an initial segment of p'. Of course, this extension-relation is a partial ordering[4] of the set of all positions, that is, if p' extends p and p'' extends p', then p'' extends p, and if p and p' extend each other, then $p = p'$. The empty sequence $\emptyset$ is the smallest element, and the plays $(\bar{x}; \bar{y})$ are

[4] See Example 5.7 for the definition of partial order. Indeed this is a tree-ordering. See Example 5.8 for the definition of tree-ordering.

maximal elements of this partial ordering. A common problem of games is that the set of all positions is huge.

A *strategy of player* **I** *in position* $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\sigma = (\sigma_0, \ldots, \sigma_{n-1-i})$$

of functions $\sigma_j : A^j \to A$. We say that player **I** has *used strategy* σ *after position* p *in the play* $(\bar{x}; \bar{y})$, if $(\bar{x}; \bar{y})$ extends p and for all j with $i < j < n$ we have

$$x_j = \sigma_{j-i}(y_i, \ldots, y_{j-1})$$

and

$$x_i = \sigma_0.$$

The strategy σ of player **I** in position p is a *winning strategy in position* p, if every play extending p where player **I** has used σ after position p is a win for player **I**.

A *strategy of player* **II** *in position* p in the game $\mathcal{G}_n(A, W)$ is a sequence

$$\tau = (\tau_0, \ldots, \tau_{n-1-i})$$

of functions $\tau_j : A^{j+1} \to A$. We say that player **II** has *used strategy* τ *after position* p *in the play* $(\bar{x}; \bar{y})$ if $(\bar{x}; \bar{y})$ extends p and for all j with $i \leq j < n$ we have

$$y_j = \tau_{j-i}(x_i, \ldots, x_j).$$

The strategy τ of player **II** in position p is a *winning strategy in position* p, if every play extending p where player **II** has used τ after p is a win for player **II**.

The following important lemma shows that if player **II** has a chance in the beginning, i.e. player **I** does not already have a winning strategy, she has a chance all the way.

Lemma 3.7 (Survival Lemma) *Suppose A is a set, n is a natural number, $W \subseteq A^{2n}$ and $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in the game $\mathcal{G}_n(A, W)$, with $i < n$. Suppose furthermore that player* **I** *does not have a winning strategy in position p. Then for every $x_i \in A$ there is $y_i \in A$ such that player* **I** *does not have a winning strategy in position $p' = (x_0, y_0, \ldots, x_i, y_i)$.*

Proof The proof is by contradiction. The intuition is clear: if player **I** had a smart move x_i so that he has a strategy for winning whatever the response y_i of player **II** is, then we could argue that, contrary to the hypothesis, player **I** had

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a winning strategy already in position p, as he wins whatever **II** moves. Let us now make this idea more exact. Suppose there were an $x_i \in A$ such that for all $y_i \in A$ player **I** has a winning strategy σ^{y_i} in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. We define a strategy $\sigma = (\sigma_0, \ldots, \sigma_{n-1-i})$ of player **I** in position p as follows: $\sigma_0(\emptyset) = x_i$ and

$$\sigma_{j-i}(y_i, \ldots, y_{j-i}) = \sigma^{y_i}(y_{i+1}, \ldots, y_{j-i}).$$

This is a winning strategy of **I** in position p, contrary to our assumption that none exists. □

The following concept is of fundamental importance in game theory and in applications to logic, in particular:

Definition 3.8 A game is called *determined* if one of the players has a winning strategy. Otherwise the game is *non-determined*.

Virtually all games that one comes across in logic are determined. The following theorem is the crucial fact behind this phenomenon:

Theorem 3.9 (Zermelo) *If A is any set, n is a natural number, and $W \subseteq A^{2n}$, then the game $\mathcal{G}_n(A, W)$ is determined.*

Proof Suppose player **I** has no winning strategy. Then player **II** has a winning strategy based on repeated use of Lemma 3.7. Player **II** notes that in the beginning of the game, that is, in position $\emptyset$, player **I** does not have a winning strategy. Then by the Survival Lemma 3.7 she can, whatever player **I** moves, find a move such that afterwards player **I** still does not have a winning strategy. In short, the strategy of player **II** is to prevent player **I** from having a winning strategy. After n rounds the game ends and player **I** still does not have a winning strategy. That means player **I** has lost and player **II** has won. Let us now make this more precise: We define a strategy

$$\tau = (\tau_0, \ldots, \tau_{n-1})$$

of player **II** in the game $\mathcal{G}_n(A, W)$ as follows: Let a be some arbitrary element of A. By Lemma 3.7 we have for each position $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ in the game $\mathcal{G}_n(A, W)$ such that player **I** does not have a winning strategy in position p and each $x_i \in A$ some $y_i \in A$ such that player **I** does not have a winning strategy in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. Let us denote this y_i by

$$y_i = f(p, x_i).$$

If $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in which player **I** *does* have a winning strategy, we let $f(p, x_i) = a$. We have defined a function f defined on positions p and elements $x_i \in A$. Let $\tau_0(x_0) = f(\emptyset, x_0)$. Assuming $\tau_0, \ldots, \tau_{i-1}$ have been defined already, let

$$\tau_i(x_0, \ldots, x_i) = f(p, x_i),$$

where

$$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$$

and

$$\begin{aligned} y_0 &= \tau_0(x_0) \\ y_{i-1} &= \tau_{i-1}(x_0, \ldots, x_{i-1}). \end{aligned}$$

It is easy to see that in every play in which player **II** uses this strategy, every position p is such that player **I** *does not* have a winning strategy in position p. It is also easy to see that this is a winning strategy of player **II**. □

3.5 Infinite Games

The concept of a game is by no means limited to games with just finitely many rounds. Imagine a chess board which extends the usual board left and right without end. Then the chess game could go on for infinitely many rounds without the same configuration of pieces coming up twice. A simple infinite game is one in which two players pick natural numbers each choosing a bigger number, if he or she can, than the opponent. There is no end to this game, since there are infinitely many natural numbers. A third kind of infinite game is the following:

Example 3.10 Suppose A is a set of real numbers on the unit interval. We describe a game we denote by $G(A)$. During the game the players decide the decimal expansion of a real number $r = 0.d_0d_1\ldots$ on the interval $[0, 1]$. Player **I** decides the even digits d_{2n} and player **II** the odd digits d_{2n+1}. Player **II** wins if $r \in A$. If A is countable, say $A = \{b_n : n \in \mathbb{N}\}$, player **I** has a winning strategy: during round n he chooses the digit d_{2n} so that $r \neq b_n$. If the complement of A is countable, player **II** wins with the same strategy. What if A and its complement are uncountable? This is a well-known and much studied hard question. (See e.g. Jech (1997).)

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I	II
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Figure 3.5 An infinite game.

If A is any set, we use $A^{\mathbb{N}}$ to denote infinite sequences

$$(x_0, x_1, \ldots)$$

of elements of A. We can think of such sequences as limits of an increasing sequence

$$(x_0), (x_0, x_1), (x_0, x_1, x_2), \ldots$$

of finite sequences.

Let A be an arbitrary set. Let $W \subseteq A^{\mathbb{N}}$. We define the game

$$\mathcal{G}_\omega(A, W)$$

as follows (see Figure 3.5): An infinite sequence

$$(\bar{x}; \bar{y}) = (x_0, y_0, x_1, y_1, \ldots),$$

of elements of A is called a *play* (of $\mathcal{G}_\omega(A, W)$). A *play* of one of the players is likewise any infinite sequence $\bar{x} = (x_0, x_1, \ldots)$ of elements of A. The play $(\bar{x}; \bar{y})$ is a *win for* player **II** if

$$(x_0, y_0, x_1, y_1, \ldots) \in W$$

and otherwise a *win for* player **I** .

A *strategy of* player **I** in the game $\mathcal{G}_\omega(A, W)$ is an infinite sequence

$$\sigma = (\sigma_0, \sigma_1, \ldots)$$

of functions $\sigma_i : A^i \to A$. We say that player **I** has *used the strategy* σ *in the play* $(\bar{x}; \bar{y})$ if for all $i \in \mathbb{N}$:

$$x_i = \sigma_i(y_0, \ldots, y_{i-1})$$

and

$$x_0 = \sigma_0.$$

The strategy σ of player **I** is a *winning strategy*, if every play where **I** has used σ is a win for player **I**.

A *strategy of* player **II** in the game $\mathcal{G}_\omega(A, W)$ is an infinite sequence

$$\tau = (\tau_0, \tau_1, \ldots)$$

of functions $\tau_i : A^{i+1} \to A$. We say that player **II** has *used the strategy* τ *in the play* $(\bar{x}; \bar{y})$ if for all $i < n$:

$$y_i = \tau_i(x_0, \ldots, x_i).$$

The strategy τ of player **II** is a *winning strategy*, if every play where player **II** has used τ is a win for player **II**. A player is said to *win the game* $\mathcal{G}_\omega(A, W)$ if he or she has a winning strategy in it.

A *position* of the infinite game $\mathcal{G}_\omega(A, W)$ is any initial segment

$$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$$

of a play $(\bar{x}; \bar{y})$. We say that player **I** has *used strategy* $\sigma = (\sigma_0, \sigma_1, \ldots)$ *after position* p *in the play* $(\bar{x}; \bar{y})$, if $(\bar{x}; \bar{y})$ extends p and for all j with $i < j$ we have $x_j = \sigma_{j-i}(y_i, \ldots, y_{j-1})$ and $x_i = \sigma_0$. The strategy σ of player **I** is a *winning strategy in position* p, if every play extending p where player **I** has used σ after position p is a win for player **I**. We say that player **II** has *used strategy* $\tau = (\tau_0, \tau_1, \ldots)$ *after position* p *in the play* $(\bar{x}; \bar{y})$ if for all j with $i \leq j$ we have $y_j = \tau_{j-i}(x_i, \ldots, x_j)$. The strategy τ of player **II** is a *winning strategy in position* p, if every play extending p where player **II** has used τ after p is a win for player **II**.

An important example of a class of infinite games is the class of *open* or *closed* games of length ω. A subset W of $A^{\mathbb{N}}$ is *open,*[5] if

$$(x_0, y_0, x_1, y_1, \ldots) \in W$$

implies the existence of $n \in \mathbb{N}$ such that

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x'_n, y'_n, x'_{n+1}, y'_{n+1}, \ldots) \in W$$

for all $x'_n, y'_n, x'_{n+1}, y'_{n+1}, \ldots \in A$. Respectively, W is *closed* if $A^{\mathbb{N}} \setminus W$ is open. Finally, W is *clopen* if it is both open and closed. We call a game $G_\omega(A, W)$ closed (or open or clopen) if the set W is. We are mainly concerned in this book with closed games. A typical strategy of player **II** in a closed game is to “hang in there”, as she knows that if player **I** ends up winning the play $p = (x_0, y_0, \ldots)$, that is, $p \notin W$, there is some n such that player **I** won the game already in position $(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$.

[5] The collection of open subsets of $A^{\mathbb{N}}$ is a topology, hence the name.

We can think of infinite games as *limits* of finite games as follows: Any finite game $G_n(A, W)$ can be made infinite by disregarding the moves after the usual n moves. The resulting infinite game is clopen (see Exercise 3.31). On the other hand, if $G_\omega(A, W)$ is an infinite game and $n \in \mathbb{N}$ we can form an n-round game by simply considering only the first n rounds of $G_\omega(A, W)$ and declaring a play of n rounds a win for player **II** if *any* infinite play extending it is in W. Unless W is open or closed, there may be very little connection between the resulting finite games and the original infinite game (see however Exercise 3.32).

Lemma 3.11 (Infinite Survival Lemma) *Suppose A is a set, $W \subseteq A^{\mathbb{N}}$, and $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in the game $\mathcal{G}_\omega(A, W)$, with $i \in \mathbb{N}$. Suppose furthermore that player* **I** *does not have a winning strategy in position p. Then for every $x_i \in A$ there is $y_i \in A$ such that player* **I** *does not have a winning strategy in position $p' = (x_0, y_0, \ldots, x_i, y_i)$.*

Proof The proof is by contradiction. Suppose there were an $x_i \in A$ such that for all $y_i \in A$ player **I** has a winning strategy σ^{y_i} in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. We define a strategy $\sigma = (\sigma_0, \sigma_1, \ldots)$ of player **I** in position p as follows: $\sigma_0(\emptyset) = x_i$ and for $j > i$,

$$\sigma_{j-i}(y_i, \ldots, y_{j-1}) = \sigma^{y_i}(y_{i+1}, \ldots, y_{j-i}).$$

This is a winning strategy of player **I** in position p, contrary to assumption. □

Theorem 3.12 (Gale–Stewart) *If A is any set and $W \subseteq A^{\mathbb{N}}$ is open or closed, then the game $\mathcal{G}_\omega(A, W)$ is determined.*

Proof Suppose first W is closed and player **I** has no winning strategy. We define a strategy

$$\tau = (\tau_0, \tau_1, \ldots)$$

of player **II** in the game $\mathcal{G}_\omega(A, W)$ as follows: Let a be some arbitrary element of A. By Lemma 3.11 we have for each position $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ in the game $\mathcal{G}_\omega(A, W)$ such that player **I** does not have a winning strategy in position p, and each $x_i \in A$, some $y_i \in A$ such that player **I** does not have a winning strategy in position $p' = (x_0, y_0, \ldots, x_i, y_i)$. Let us denote this y_i by

$$y_i = f(p, x_i).$$

If $p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ is a position in which player **I** *does* have a winning strategy, we let $f(p, x_i) = a$. We have defined a function f defined on positions p and elements $x_i \in A$. Let $\tau_0(x_0) = f(\emptyset, x_0)$. Assuming $\tau_0, \ldots, \tau_{i-1}$ have been defined already, let $\tau_i(x_0, \ldots, x_i) = f(p, x_i)$, where

$p = (x_0, y_0, \ldots, x_{i-1}, y_{i-1})$ and $y_0 = \tau_0(x_0), y_{i-1} = \tau_{i-1}(x_0, \ldots, x_{i-1})$. It is easy to see that in every play in which player **II** uses this strategy, every position p is such that player **I** does not have a winning strategy in position p. It is also easy to see that this is a winning strategy of player **II**.

The proof is similar if W is open. It follows that $G_\omega(A, W)$ is determined. □

Theorem 3.12 can been vastly generalized, see e.g. (Jech, 1997, Chapter 33). The *Axiom of Determinacy* says that the game $G_\omega(A, W)$ is determined for all sets A and W. However, this axiom contradicts the Axiom of Choice. By using the Axiom of Choice one can show that there are sets A of real numbers such that the game $G(A)$ is not determined (see Exercise 3.37).

3.6 Historical Remarks and References

The mathematical theory of games was started by von Neumann and Morgenstern (1944). For the early history of two-person zero-sum games of perfect information, see Schwalbe and Walker (2001). See Mycielski (1992) for a more recent survey on games of perfect information. Theorem 3.12 goes back to Gale and Stewart (1953).

Exercises

3.1 Consider the following game: Player **I** picks a natural number n. Then player **II** picks a natural number m. If $2^m = n$, then **II** wins, otherwise **I** wins. Express this game in the form $\mathcal{G}_1(A, W)$.

3.2 Consider the following game: Player **I** picks a natural number n. Then player **II** picks two natural numbers m and k. If $m \cdot k = n$, then **II** wins, otherwise **I** wins. Express this game in the form $\mathcal{G}_2(A, W)$.

3.3 Consider $\mathcal{G}_3(A, W)$, where $A = \{0, 1, 2\}$ and

1. $W = \{(x_0, y_0, x_1, y_1, x_2, y_2) \in A^3 : x_0 = y_2\}$.
2. $W = \{(x_0, y_0, x_1, y_1, x_2, y_2) \in A^3 : y_0 \neq x_2$ or $y_2 \neq x_0\}$.
3. $W = \{(x_0, y_0, x_1, y_1, x_2, y_2) \in A^3 : x_0 \neq y_2$ and $x_1 \neq y_2$ and $x_2 \neq y_2\}$.

Who has a winning strategy?

3.4 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a mapping. Express the condition that f is uniformly continuous as a game and as the truth of a first-order sentence in a suitable structure.

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3.10 Examine the game determined by condition (3.1) $M = \mathbb{N}$ and $W^{\mathcal{M}} = \{(a_0, b_0, a_1, b_1) \in M^4 : a_0 < b_0$ and either a_1 does not divide b_0 or $b_1 = a_1 = 1$ or $b_1 = a_1 = b_0\}$. Who has a winning strategy?

3.11 Suppose X is a set of positions of the game $G_n(A, W)$ such that

1. $\emptyset \in X$.
2. For all $i < n$, all $(x_0, y_0, \ldots, x_{i-1}, y_{i-1}) \in X$, and all $x_i \in A$ there is $y_i \in A$ such that $(x_0, y_0, \ldots, x_i, y_i) \in X$.
3. If $p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in X$, then $p \in W$.

Show that player **II** has a winning strategy in the game $G_n(A, W)$. Give such a set for the game of Example 3.1.

3.12 Suppose that player **II** has a winning strategy in the game $G_n(A, W)$. Show that there is a set X of positions of the game $G_n(A, W)$ satisfying conditions 1–3 of the previous exercise.

3.13 Suppose X is a set of positions of the game $G_n(A, W)$ such that

1. $\emptyset \in X$.
2. For all $i < n$, all $(x_0, y_0, \ldots, x_{i-1}, y_{i-1}) \in X$ there is $x_i \in A$ such that for all $y_i \in A$ we have $(x_0, y_0, \ldots, x_i, y_i) \in X$.
3. If $p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in X$, then $p \notin W$.

Show that player **I** has a winning strategy in the game $G_n(A, W)$. Give such a set for the game of Example 3.1 when we start with seven tokens.

3.14 Suppose that player **I** has a winning strategy in the game $G_n(A, W)$. Show that there is a set X of positions of the game $G_n(A, W)$ satisfying conditions 1–3 of the previous exercise.

3.15 Suppose A is finite. Describe an algorithm which searches for a winning strategy for a player in $\mathcal{G}_n(A, W)$, provided the player has one.

3.16 Finish the proof of Lemma 3.7 by showing that the strategy described in the proof is indeed a winning strategy of player **I**.

3.17 Finish the proof of Theorem 3.9 by showing that the strategy described in the proof is indeed a winning strategy of player **II**.

3.18 Consider $\mathcal{G}_2(A, W)$, where $A = \{0, 1\}$ and

1. $W = \{(x_0, y_0, x_1, y_1) \in A^2 : x_0 = y_1\}$.
2. $W = \{(x_0, y_0, x_1, y_1) \in A^2 : y_0 \neq x_1$ or $y_1 \neq x_0\}$.
3. $W = \{(x_0, y_0, x_1, y_1) \in A^2 : x_0 \neq y_1$ and $x_1 \neq y_1\}$.

In each case give a winning strategy for one of the players.

3.19 Suppose σ is a strategy of player **I** and τ a strategy of player **II** in $\mathcal{G}_n(A, W)$. Show that there is exactly one play $(\bar{x}; \bar{y})$ of $\mathcal{G}_n(A, W)$ such that player **I** has used σ and player **II** has used τ in it.

3.20 Show that at most one player can have a winning strategy in $\mathcal{G}_n(A, W)$.

5

Models

5.1 Introduction

The concept of a model (or structure) is one of the most fundamental in logic. In brief, while the meaning of logical symbols $\wedge, \vee, \exists, \ldots$ is always fixed, models give meaning to non-logical symbols such as constant, predicate, and function symbols. When we have agreed about the meaning of the logical and non-logical symbols of logic, we can then define the meaning of arbitrary formulas.

Depending on context and preference, models appear in logic in two roles. They can serve the auxiliary role of clarifying logical derivation. For example, one quick way to tell what it means for φ to be a logical consequence of ψ is to say that in every model where ψ is true also φ is true. It is then an almost trivial matter to understand why for example $\forall x \exists y \varphi$ is a logical consequence of $\exists y \forall x \varphi$ but $\forall y \exists x \varphi$ is in general not.

Alternatively models can be the prime objects of investigation and it is the logical derivation that is in an auxiliary role of throwing light on properties of models. This is manifestly demonstrated by the Completeness Theorem which says that any set T of first-order sentences has a model unless a contradiction can be logically derived from T, which entails that the two alternative perspectives of models are really equivalent. Since derivations are finite, this implies the important Compactness Theorem: If a set of first-order sentences is such that each of its finite subsets has a model it itself has a model. The Compactness Theorem has led to an abundance of non-isomorphic models of first-order theories, and constitutes the origin of the whole subject of Model Theory. In this chapter models are indeed the prime objects of investigation and we introduce auxiliary concepts such as the Ehrenfeucht–Fraïssé Game that help us understand models.

We use the words "model" and "structure" as synonyms. We have a slight

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preference for the word "structure" in a context where absolute generality prevails and the structures are not assumed to satisfy any particular axioms. Respectively, our preference is to call a structure that satisfies some given axioms a model, so a structure satisfying a theory is called a model of the theory.

5.2 Basic Concepts

A *vocabulary* is any set L of predicate symbols $P, Q, R, \ldots$, function symbols $f, g, h, \ldots$, and constant symbols $c, d, e, \ldots$. Each vocabulary has an *arity-function*

$$\#_L : L \to \mathbb{N}$$

which tells the arity of each symbol. Thus if $P \in L$, then P is a $\#_L(P)$-ary predicate symbol. If $f \in L$, then f is a $\#_L(f)$-ary function symbol. Finally, $\#_L(c)$ is assumed to be 0 for constants $c \in L$. Predicate or function symbols of arity 1 are called *unary* or *monadic*, and those of arity 2 are called *binary*. A vocabulary is called unary (or binary) if it contains only unary (respectively, binary) symbols. A vocabulary is called *relational* if it contains no function or constant symbols.

Definition 5.1 An L*-structure* (or L*-model*) is a pair $\mathcal{M} = (M, \mathrm{Val}_{\mathcal{M}})$, where M is a non-empty set called the *universe (or the domain)* of $\mathcal{M}$, and $\mathrm{Val}_{\mathcal{M}}$ is a function defined on L with the following properties:

1. If $R \in L$ is a relation symbol and $\#_L(R) = n$, then $\mathrm{Val}_{\mathcal{M}}(R) \subseteq M^n$.
2. If $f \in L$ is a function symbol and $\#_L(f) = n$, then $\mathrm{Val}_{\mathcal{M}}(f) : M^n \to M$.
3. If $c \in L$ is a constant symbol, then $\mathrm{Val}_{\mathcal{M}}(c) \in M$.

We use $\mathrm{Str}(L)$ to denote the class of all L-structures.

We usually shorten $\mathrm{Val}_{\mathcal{M}}(R)$ to $R^{\mathcal{M}}$, $\mathrm{Val}_{\mathcal{M}}(f)$ to $f^{\mathcal{M}}$, and $\mathrm{Val}_{\mathcal{M}}(c)$ to $c^{\mathcal{M}}$. If no confusion arises, we use the notation

$$\mathcal{M} = (M, R_1^{\mathcal{M}}, \ldots, R_n^{\mathcal{M}}, f_1^{\mathcal{M}}, \ldots, f_m^{\mathcal{M}}, c_1^{\mathcal{M}}, \ldots, c_k^{\mathcal{M}})$$

for an L-structure $\mathcal{M}$, where $L = \{R_1, \ldots, R_n, f_1, \ldots, f_m, c_1 \ldots, c_k\}$.

Example 5.2 Graphs are L-structures for the relational vocabulary $L = \{E\}$, where E is a predicate symbol with $\#_L(E) = 2$. Groups are L-structures for $L = \{\circ\}$, where $\circ$ is a binary function symbol. Fields are L-structures for $L = \{+, \cdot, 0, 1\}$, where $+, \cdot$ are binary function symbols and $0, 1$ are constant symbols. Ordered sets (i.e. linear orders) are L-structures for the relational vocabulary $L = \{<\}$, where $<$ is a binary predicate symbol. If $L = \emptyset$, an L-structure (M) is a structure with just the universe and no structure in it.

If $\mathcal{M}$ is a structure and π maps M bijectively onto another set M', we can use π to copy the relations, functions, and constants of $\mathcal{M}$ on M'. In this way we get a perfect copy $\mathcal{M}'$ of $\mathcal{M}$ which differs from $\mathcal{M}$ only in the respect that the underlying elements are different. We then say that $\mathcal{M}'$ is an isomorphic copy of $\mathcal{M}$. For all practical purposes we consider the structures $\mathcal{M}$ and $\mathcal{M}'$ as one and the same structure. However, they are not the same structure, just isomorphic. This may sound as if isomorphism was a rather trivial matter, but this is not true. In many cases it is a highly non-trivial enterprise to investigate whether two structures are isomorphic or not. In the realm of finite structures the question of deciding whether two given structures are isomorphic or not is a famous case of a complexity question which is between P (polynomial time) and NP (non-deterministic polynomial time) and about which we do not know whether it is NP-complete. In the light of present knowledge it is conceivable that this question is strictly between P and NP.

Definition 5.3 L-structures $\mathcal{M}$ and $\mathcal{M}'$ are *isomorphic* if there is a bijection

$$\pi : M \to M'$$

such that

1. For all $a_1, \ldots, a_{\#_L(R)} \in M$:
$$(a_1, \ldots, a_{\#_L(R)}) \in R^{\mathcal{M}} \iff (\pi(a_1), \ldots, \pi(a_{\#_L(R)})) \in R^{\mathcal{M}'}.$$
2. For all $a_1, \ldots, a_{\#_L(f)} \in M$:
$$f^{\mathcal{M}'}(\pi(a_1), \ldots, \pi(a_{\#_L(f)})) = \pi(f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)})).$$
3. For all $c \in L$: $\pi(c^{\mathcal{M}}) = c^{\mathcal{M}'}$.

In this case we say that π is an *isomorphism* $\mathcal{M} \to \mathcal{M}'$, denoted

$$\pi : \mathcal{M} \cong \mathcal{M}'.$$

If also $\mathcal{M} = \mathcal{M}'$, we say that π is an *automorphism* of $\mathcal{M}$.

Example 5.4 *Unary* (or *monadic*) structures, i.e. L-structures for unary L, are particularly simple and easy to deal with. Figure 5.1 depicts a unary structure. Suppose L consists of unary predicate symbols $R_1, \ldots, R_n$ and $\mathcal{A}$ is an L-structure. If $X \subseteq A$ and $d \in \{0, 1\}$, let $X^d = X$ if $d = 0$ and $X^d = A \setminus X$

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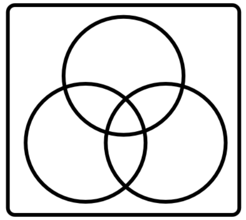

Figure 5.1 A unary structure.

otherwise. Suppose $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$. The *$\epsilon$-constituent* of $\mathcal{A}$ is the set

$$C_\epsilon(\mathcal{A}) = \bigcap_{i=1}^{n} (R_i^{\mathcal{A}})^{\epsilon(i)}.$$

A priori, the 2^n sets $C_\epsilon(\mathcal{A})$ can each have any cardinality whatsoever. It is the nature of unary structures that the constituents are totally independent of each other. If $\mathcal{A} \cong \mathcal{B}$, then

$$|C_\epsilon(\mathcal{A})| = |C_\epsilon(\mathcal{B})| \tag{5.1}$$

for every ϵ. Conversely, if two L-structures $\mathcal{A}$ and $\mathcal{B}$ satisfy Equation (5.1) for every ϵ, then $\mathcal{A} \cong \mathcal{B}$ (see Exercise 5.6). We can say that the function $\epsilon \mapsto |C_\epsilon(\mathcal{A})|$ characterizes completely (i.e. up to isomorphism) the unary structure $\mathcal{A}$. There is nothing more we can say about $\mathcal{A}$ but this function.

Example 5.5 *Equivalence relations*, i.e. L-structures $\mathcal{M}$ for $L = \{\sim\}$ such that $\sim^{\mathcal{M}}$ is a symmetric ($x \sim y \Rightarrow y \sim x$), transitive ($x \sim y \sim z \Rightarrow x \sim z$), and reflexive ($x \sim x$) relation on M can be characterized almost as easily as unary structures. Let for every cardinal number $\kappa \leq |M|$ the number of equivalence classes of $\sim^{\mathcal{M}}$ of cardinality κ be denoted by $EC_\kappa(\mathcal{M})$. If $\mathcal{A} \cong \mathcal{B}$, then

$$EC_\kappa(\mathcal{A}) = EC_\kappa(\mathcal{B}) \tag{5.2}$$

for every $\kappa \leq |A|$. Conversely, if two L-structures $\mathcal{A}$ and $\mathcal{B}$ satisfy Equation (5.2) for every $\kappa \leq |A \cup B|$, then $\mathcal{A} \cong \mathcal{B}$ (see Exercise 5.12). We can say that the function $\kappa \mapsto EC_\kappa(\mathcal{A})$ characterizes completely (i.e. up to isomorphism) the equivalence relation $\mathcal{A}$. There is nothing more we can say about $\mathcal{A}$ but this function. For equivalence relations on a finite universe of size n this

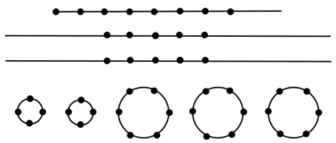

Figure 5.4 A successor structure.

If $\mathcal{M}$ is a successor structure, let $Cmp_{\mathcal{M}}$ be the set of components of $\mathcal{M}$ and

$$CC_n(\mathcal{M}) = |\{C \in Cmp_{\mathcal{M}} : C \text{ is an } n\text{-cycle component}\}|,$$

$$CC_\infty(\mathcal{M}) = |\{C \in Cmp_{\mathcal{M}} : C \text{ is a } \mathbb{Z}\text{-component}\}|.$$

Two successor structures $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $CC_a(\mathcal{M}) = CC_a(\mathcal{N})$ for all $a \in \mathbb{N} \cup \{\infty\}$.

5.3 Substructures

The concept of a substructure is in principle a very simple one, especially for relational vocabularies. There are however subtleties which deserve special attention when function symbols are involved.

Definition 5.10 An L-structure $\mathcal{M}$ is a *substructure* of another L-structure $\mathcal{M}'$, in symbols $\mathcal{M} \subseteq \mathcal{M}'$, if:

1. $M \subseteq M'$.
2. $R^{\mathcal{M}} = R^{\mathcal{M}'} \cap M^n$ if $R \in L$ is an n-ary predicate symbol.
3. $f^{\mathcal{M}} = f^{\mathcal{M}'} \restriction M^n$ if $f \in L$ is an n-ary function symbol.
4. $c^{\mathcal{M}} = c^{\mathcal{M}'}$ if $c \in L$ is a constant symbol.

Substructures are particularly easy to understand in the case that L is relational. Then any subset M of an L-structure $\mathcal{M}'$ determines a substructure $\mathcal{M}$ the universe of which is M. If L is not relational we have to worry about the question whether M is closed under the functions $f^{\mathcal{M}'}$, $f \in L$, and whether the interpretations $c^{\mathcal{M}'}$ of constant symbols $c \in L$ are in M. For example, if $L = \{f\}$ where f is a unary function symbol, then any substructure of an L-structure which contains an element a has to contain also $f^{\mathcal{M}'}(a), f^{\mathcal{M}'}(f^{\mathcal{M}'}(a))$, etc. A substructure of a group need not be a subgroup

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even when it is closed under the group operation. For example, $(\mathbb{N}, +)$ is a substructure of $(\mathbb{Z}, +)$ but it is not a group. A substructure of a linear order is again a linear order. Similarly, a substructure of a partial order is again a partial order. A substructure of a tree is a tree if it has a smallest element.

Lemma 5.11 *Suppose L is a vocabulary, $\mathcal{M}$ an L-structure, and $X \subseteq M$. Suppose furthermore that either L contains constant symbols or $X \neq \emptyset$. There is a unique L-structure $\mathcal{N}$ such that:*

1. $\mathcal{N} \subseteq \mathcal{M}$.
2. $X \subseteq N$.
3. If $\mathcal{N}' \subseteq \mathcal{M}$ and $X \subseteq N'$, then $\mathcal{N} \subseteq \mathcal{N}'$.

Proof Let $X_0 = X \cup \{c^{\mathcal{M}} : c \in L\}$ and inductively

$$X_{n+1} = X_n \cup \{f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)}) : a_1, \ldots, a_{\#_L(f)} \in X_n, f \in L\}.$$

It is easy to see that the set $N = \bigcup_{n \in \mathbb{N}} X_n$ is the universe of the unique structure $\mathcal{N}$ claimed to exist in the lemma. □

We call the unique structure $\mathcal{N}$ of Lemma 5.11 the substructure of $\mathcal{M}$ *generated* by X and denote it by $[X]_{\mathcal{M}}$. The following lemma is used repeatedly in the sequel.

Lemma 5.12 *Suppose L is a vocabulary. Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $\pi : M \to N$ is a partial mapping. There is at most one isomorphism $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{N}}$ extending π.*

5.4 Back-and-Forth Sets

One of the main themes of this book is the question: Given two structures $\mathcal{M}$ and $\mathcal{N}$, how do we measure how close they are to being isomorphic? They may be non-isomorphic for a totally obvious reason, e.g. two graphs one of which has a triangle while the other does not. They may also be non-isomorphic for an extremely subtle reason which involves the use of the Axiom of Choice (see e.g. Lemma 9.9). One of the basic tools in trying to answer this question is the concept of partial isomorphism.

Definition 5.13 Suppose L is a vocabulary and $\mathcal{M}, \mathcal{M}'$ are L-structures. A partial mapping $\pi : M \to M'$ is a *partial isomorphism* $\mathcal{M} \to \mathcal{M}'$ if there is an isomorphism $\pi^* : [\mathrm{dom}(\pi)]_{\mathcal{M}} \to [\mathrm{rng}(\pi)]_{\mathcal{M}'}$ extending π. We use $\mathrm{Part}(\mathcal{M}, \mathcal{M}')$ to denote the set of partial isomorphisms $\mathcal{M} \to \mathcal{M}'$. If $\mathcal{M} = \mathcal{M}'$ we call π a *partial automorphism*.

Note that the extension π^* referred to in Definition 5.13 is by Lemma 5.12 necessarily unique.

The main topic of this section, the back-and-forth sets, are very useful weaker versions of isomorphisms. To get a picture of this, suppose $f : \mathcal{A} \cong \mathcal{B}$. Then $f \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ and we can go back and forth between $\mathcal{A}$ and $\mathcal{B}$ with f in the following sense:

$$\forall a \in A \exists b \in B(f(a) = b) \quad (5.6)$$

$$\forall b \in B \exists a \in A(f(a) = b). \quad (5.7)$$

We now generalize this to a situation where we do not quite have an isomorphism but only a set P which reflects the back and forth conditions (5.8) and (5.9) of an isomorphism.

Definition 5.14 Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures. A *back-and-forth set* for $\mathcal{A}$ and $\mathcal{B}$ is any non-empty set $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B})$ such that

$$\forall f \in P \forall a \in A \exists g \in P(f \subseteq g \text{ and } a \in \mathrm{dom}(g)) \quad (5.8)$$

$$\forall f \in P \forall b \in B \exists g \in P(f \subseteq g \text{ and } b \in \mathrm{rng}(g)). \quad (5.9)$$

The structures $\mathcal{A}$ and $\mathcal{B}$ are said to be *partially isomorphic*, in symbols $\mathcal{A} \simeq_p \mathcal{B}$, if there is a back-and-forth set for them.

Lemma 5.15 *The relation $\simeq_p$ is an equivalence relation on $\mathrm{Str}(L)$.*

Proof The relation $\simeq_p$ is reflexive, because $\{id_A\}$ is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. If P is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$, then $\{f^{-1} : f \in P\}$ is a back-and-forth set for $\mathcal{B}$ and $\mathcal{A}$. Finally, if P_1 is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$ and P_2 is a back-and-forth set for $\mathcal{B}$ and $\mathcal{C}$, then $\{f_2 \circ f_1 : f_1 \in P_1, f_2 \in P_2\}$ is a back-and-forth set for $\mathcal{A}$ and $\mathcal{C}$, where we stipulate $\mathrm{dom}(f_2 \circ f_1) = f_1^{-1}(\mathrm{dom}(f_2))$. □

Proposition 5.16 *If $\mathcal{A} \simeq_p \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are countable, then $\mathcal{A} \cong \mathcal{B}$.*

Proof Let us enumerate A as $(a_n : n < \omega)$ and B as $(b_n : n < \omega)$. Let P be a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. Since $P \neq \emptyset$, there is some $f_0 \in P$. We define a sequence $(f_n : n < \omega)$ of elements of P as follows: Suppose $f_n \in P$ is defined. If n is even, say $n = 2m$, let $y \in B$ and $f_{n+1} \in P$ such that $f_n \cup \{(a_m, y)\} \subseteq f_{n+1}$. If n is odd, say $n = 2m + 1$, let $x \in A$ and $f_{n+1} \in P$ such that $f_n \cup \{(x, b_m)\} \subseteq f_{n+1}$. Finally, let

$$f = \bigcup_{n=0}^{\infty} f_n.$$

Clearly, $f : \mathcal{A} \cong \mathcal{B}$. □

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This proposition is not true for uncountable structures. Indeed, let $L = \emptyset$ and let $\mathcal{A}$ and $\mathcal{B}$ be any infinite L-structures. Then there is a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$ (Exercise 5.28). Thus $\mathcal{A} \simeq_p \mathcal{B}$. But $\mathcal{A} \not\cong \mathcal{B}$ if, for example, $A = \mathbb{Q}$ and $B = \mathbb{R}$. The failure of Proposition 5.16 to generalize is a major topic in the sequel.

Proposition 5.17 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are dense linear orders without endpoints. Then $\mathcal{A} \simeq_p \mathcal{B}$.*

Proof Let $P = \{f \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(f) \text{ is finite}\}$. It turns out that this straightforward choice works. Clearly, $P \neq \emptyset$. Suppose then $f \in P$ and $a \in A$. Let us enumerate f as $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ where $a_1 < \ldots < a_n$. Since f is a partial isomorphism, also $b_1 < \ldots < b_n$. Now we consider different cases. If $a < a_1$, we choose $b < b_1$ and then $f \cup \{(a, b)\} \in P$. If $a_i < a < a_{i+1}$, we choose $b \in B$ so that $b_i < b < b_{i+1}$ and then $f \cup \{(a, b)\} \in P$. If $a_n < a$, we choose $b > b_n$ and again $f \cup \{(a, b)\} \in P$. Finally, if $a = a_i$, we let $b = b_i$ and then $f \cup \{(a, b)\} = f \in P$. We have proved (5.8). Condition (5.9) is proved similarly. □

Putting Proposition 5.16 and Proposition 5.17 together yields the famous result of Cantor (1895): countable dense linear orders without endpoints are isomorphic. See Exercise 6.29 for a more general result.

5.5 The Ehrenfeucht–Fraïssé Game

In Section 4.3 we introduced the Ehrenfeucht–Fraïssé Game played on two graphs. This game was used to measure to what extent two graphs have similar properties, especially properties expressible in the first-order language of graphs limited to a fixed quantifier rank. In this section we extend this game to the context of arbitrary structures, not just graphs.

Let us recall the basic idea behind the Ehrenfeucht–Fraïssé Game. Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures for some relational L. We imagine a situation in which two mathematicians argue about whether $\mathcal{A}$ and $\mathcal{B}$ are isomorphic or not. The mathematician that we denote by **II** claims that they are isomorphic, while the other mathematician whom we call **I** claims the models have an intrinsic structural difference and they cannot possibly be isomorphic.

The matter would be quickly resolved if **II** was required to show the claimed isomorphism. But the rules of the game are different. The rules are such that **II** is required to show only small pieces of the claimed isomorphism.

More exactly, **I** asks what is the image of an element a_1 of A that he chooses